Journal of Global Optimization (2006) 35: 483–492
DOI 10.1007/s10898-006-6755-y

Multivalued Regularized Equilibrium Problems

MUHAMMAD ASLAM NOOR
Mathematics Department, COMSATS Institute of Information Technology, Islamabad, Pakistan (e-mail: noormaslam@hotmail.com)

(Received: 6 May 2003; accepted in revised form: 7 January 2006)

Abstract. In this paper, we introduce a new class of equilibrium problems known as the multivalued regularized equilibrium problems. We use the auxiliary principle technique to suggest some iterative methods for solving multivalued regularized equilibrium problems. The convergence of the proposed methods is studied under some mild conditions. As special cases, we obtain a number of known and new results for solving various classes of regularized equilibrium problems and related optimization problems.

Mathematics Subject Classifications (2000): 91B50, 49J40

Key words and phrases: Auxiliary principle, Convergence, Equilibrium problems, Proximal point algorithm, Variational inequalities

1. Introduction

Equilibrium problems theory is dynamic and is experiencing an explosive growth in both theory and applications; as a consequence, research techniques and problems are drawn from various fields. Equilibrium problems have been generalized and extended in different directions. An important and useful generalization of equilibrium problems is called the multivalued general equilibrium problems, which was introduced and studied by Noor [7]. It has been shown that a wide class of unrelated odd order and nonsymmetric free, moving, obstacle and equilibrium problems can be studied via the multivalued equilibrium problems. Almost all the results obtained for equilibrium problems are in the setting of convexity. In this paper, we consider a new class of equilibrium problems, which are called *multivalued regularized equilibrium problems*, where the convex set is replaced by the so-called uniformly prox-regular sets. The uniformly prox-regular sets are nonconvex and include the convex sets as a special case, see [2, 11]. As special cases of these new problems, we can obtain the equilibrium problems discussed in [1, 7–9]. There are several numerical methods including projection methods, Wiener-Hopf equations, descent and decomposition for solving variational inequalities. Unfortunately these methods cannot be extended for solving equilibrium problems. To overcome these difficulties, we use the auxiliary principle technique, which is mainly due to Glowinski et al. [5]. Noor [7, 8] has used this technique to suggest and analyze several numerical methods for solving various classes of equilibrium problems and optimization problems. In this paper, we show that this technique can be extended for multivalued regularized equilibrium problems. We use this technique to suggest and analyze some iterative schemes for solving multivalued regularized general equilibrium problems and study their convergence under mild conditions. As special cases, we obtain the corresponding results for equilibrium problems and variational inequalities.

2. Preliminaries

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|.\|$, respectively. Let $C(H)$ be the family of all nonempty compact subsets of H. Let $T: H \longrightarrow C(H)$ be a multivalued operator and $g: H \longrightarrow H$ be a single-valued operator. Let K be a nonempty and closed set in H. We need the following concepts from nonsmooth analysis, see [2, 11].

DEFINITION 2.1. The proximal normal cone of K at u is given by

$$N^P(K;u) := \{\xi \in H : u \in P_K[u+\alpha\xi]\},$$

where $\alpha > 0$ is a constant and

$$P_K[u] = \{u^* \in K : d_K(u) = \|u-u^*\|\}.$$

Here $d_K(.)$ is the usual distance function to the subset K, that is

$$d_K(u) = \inf_{v\in K} \|v-u\|.$$

The proximal normal cone $N^P(K;u)$ has the following characterization.

LEMMA 2.1. *Let K be a closed subset in H. Then $\zeta \in N^P(K;u)$ if and only if there exists a constant $\alpha > 0$ such that*

$$\langle\zeta, v-u\rangle \leqslant \alpha\|v-u\|^2, \quad \forall v \in K.$$

DEFINITION 2.2. The Clarke normal cone, denoted by $N^C(K;u)$, is defined as

$$N^C(K;u) = \overline{co}[N^P(K;u)],$$

where $\overline{co}$ means the closure of the convex hull. Clearly $N^P(K;u) \subset N^C(K;u)$, but the converse is not true. Note that $N^P(K;u)$ is always closed and convex, whereas $N^P(K;u)$ is convex, but may not be closed [11]. Poliquin et al. [11] and Clarke et al. [2] have introduced and studied a new class of nonconvex sets, which are called uniformly prox-regular sets. This class of uniformly prox-regular sets has played an important part in many nonconvex applications such as optimization, dynamic systems and differential inclusions. In particular, we have.

DEFINITION 2.3. For a given $r \in (0, \infty]$, a subset K is said to be uniformly r-prox-regular if and only if every nonzero proximal normal to K can be realized by an r-ball, that is, $\forall u \in K$ and $0 \neq \xi \in N^P(K;u)$, $\|\xi\| = 1$, one has

$$\langle \xi, v - u \rangle \leqslant (1/2r)\|v - u\|^2, \quad \forall v \in K.$$

It is clear that the class of uniformly prox-regular sets is sufficiently large to include the class of convex sets, p-convex sets, $C^{1,1}$ submanifolds (possibly with boundary) of H, the images under a $C^{1,1}$ diffeomorphism of convex sets and many other nonconvex sets; see [2, 11]. Note that if $r = \infty$, then uniform r-prox-regularity of K is equivalent to the convexity of K. This fact plays an important part in this paper. It is known that if K is a uniformly r-prox-regular set, then the proximal normal cone $N^P(K;u)$ is closed as a set-valued mapping. Thus, we have $N^C(K;u) = N^P(K;u)$ and take $\gamma = \frac{1}{2r}$. Clearly $\gamma = 0$, if and only if $r = \infty$.

From now onward, the set K is a uniformly prox-regular set in H, unless otherwise specified.

For a given single-valued function $F(.,.) : H \times H \longrightarrow H$, we consider the problem of finding $u \in H, g(u) \in K, \nu \in T(u)$ such that

$$F(\nu, g(v)) + \gamma\|g(v) - g(u)\|^2 \geqslant 0, \quad \forall v \in H : g(v) \in K, \tag{2.1}$$

which is called the *uniformly regularized multivalued equilibrium problem.*

If $\gamma = 0$, then the uniformly prox-regular set K becomes the convex set K and problem (2.1) is equivalent to finding $u \in H : g(u) \in K, \nu \in T(u)$ such that

$$F(\nu, g(v)) \geqslant 0, \quad \forall v \in H : g(v) \in K, \tag{2.2}$$

which is called the multivalued general equilibrium problem introduced and studied by Noor [7] using the auxiliary principle technique.

If $F(\nu, g(v)) = \langle \nu, g(v) - g(u) \rangle$, then problem (2.1) is equivalent to finding $u \in H, \nu \in T(u), g(u) \in K$ such that

$$\langle \nu, g(v) - g(u) \rangle + \gamma\|g(v) - g(u)\|^2 \geqslant 0, \quad \forall v \in H : g(v) \in K, \tag{2.3}$$

which is known as the multivalued regularized general variational inequality. It is worth mentioning that a wide class of multivalued odd order and nonsymmetric free, obstacle, moving, equilibrium and optimization problems arising in pure and applied sciences can be studied via the multivalued general variational inequalities.

In brief, for suitable and appropriate choice of the operators $T, F(.,.)$ and the space H, one can obtain several new and known classes of equilibrium problems and variational inequalities. For the applications and numerical methods of variational inequalities and equilibrium problems, see [1, 3–10, 12].

LEMMA 2.2. $\forall u, v \in H$, *we have*

$$2\langle u, v\rangle = \|u+v\|^2 - \|u\|^2 - \|v\|^2. \tag{2.4}$$

DEFINITION 2.4. $\forall u_1, u_2, z \in H, w_1 \in T(u_1), w_2 \in T(u_2)$, the bifunction $F(.,.)$ is said to be:

(i) *partially relaxed strongly g-monotone,* iff there exists a constant $\alpha > 0$, such that

$$F(w_1, g(u_2)) + F(w_2, g(z)) \leqslant \alpha \|g(z) - g(u_1)\|^2.$$

(ii) *g-monotone,* iff

$$F(w_1, g(u_2)) + F(w_2, g(u_1)) \leqslant 0.$$

(iii) *g-pseudomonotone,* iff

$$F(w_1, g(u_2)) + \gamma \|g(u_2) - g(u_1)\|^2 \geqslant 0,$$

implies

$$-F(w_2, g(u_1)) + \gamma \|g(u_2) - g(u_1)\|^2 \geqslant 0.$$

(iv) *M-Lipschitz continuous,* if and only if there exists a constant $\delta > 0$, such that

$$M(T(u_1), T(u_2)) \leqslant \delta \|u_1 - u_2\|,$$

where $M(.,.)$ is the Hausdorff metric on $C(H)$.

We remark that, if $z = u_1$, then partially relaxed strongly g-monotonicity is exactly g-monotonicity of $F(.,.)$. For $g \equiv I$, the indentity operator, Definition 2.4 reduces to the definition of partially relaxed strongly monotonicity, monotonicity and pseudomonotonicity of the bifunction $F(.,.)$.

3. Main Results

In this section, we use the auxiliary principle technique to suggest and analyze some iterative schemes for solving uniformly regularized multivalued variational inequalities.

For a given $u \in H : g(u) \in K, \nu \in T(u)$, where K a uniformly prox-regular set in H, consider the problem of finding $w \in H : g(w) \in K, \xi \in T(w)$ such that

$$F(\xi, g(v)) + \langle g(w) - g(u), g(v) - g(w)\rangle + \gamma \|g(v) - g(w)\|^2 \geqslant 0, \quad \forall g(v) \in K, \tag{3.1}$$

where $\rho > 0$ is a constant.

Inequality of type (3.1) is called the auxiliary multivalued regularized equilibrium problem. Note that if $w = u$, then w is a solution of (2.1). This simple observation enables us to suggest the following iterative method for solving (2.1).

ALGORITHM 3.1. For a given $u_0 \in K$, compute the approximate solution u_{n+1} by the iterative scheme

$$\begin{aligned} &\rho F(\eta_{n+1}, g(v)) + \langle g(u_{n+1}) - g(u_n), g(v) - g(u_{n+1})\rangle \\ &\quad + \gamma \|g(u_{n+1}) - g(u_n)\|^2 \geqslant 0, \quad \forall g(v) \in K \end{aligned} \tag{3.2}$$

$$\eta_n \in T(u_n) : ||\eta_{n+1} - \eta_n|| \leqslant M(T(u_{n+1}), T(u_n)), \quad n = 0, 1, 2, \ldots \tag{3.3}$$

Algorithm 3.1 is called the proximal point algorithm for solving uniformly regularized equilibrium problem (2.1). For suitable and appropriate choice of the operators and the spaces, one can obtain a number of new and known iterative methods for solving the equilibrium problems and variational inequalities.

We now consider the convergence analysis of Algorithm 3.1.

THEOREM 3.1. *Let* $u \in H$ *be a solution of* (3.1) *and* u_{n+1} *be the approximate solution obtained from Algorithm* 3.1. *If* $F(.,.)$ *is a g-pseudomonotone, then*

$$\begin{aligned} \{1-\gamma\}\|g(u_{n+1}) - g(u)\|^2 \leqslant \|g(u_n) - g(u)\|^2 \\ - \{1-\gamma\}\|g(u_{n+1}) - g(u_n)\|^2. \end{aligned} \tag{3.4}$$

Proof. Let $u \in H, g(u) \in K, \nu \in T(u)$ be a solution of (2.1). Then

$$F(\nu, g(v)) + \gamma \|g(v) - g(u)\|^2 \geqslant 0, \quad \forall g(v) \in K. \tag{3.5}$$

Now taking $v = u_{n+1}$ in (3.5), we have

$$F(\nu, g(u_{n+1})) + \gamma \|g(u_{n+1})) - g(u)\|^2 \geqslant 0,$$

which implies that

$$-F(\eta_{n+1}, g(u)) + \gamma \|g(u_{n+1}) - g(u)\|^2 \geqslant 0, \tag{3.6}$$

since $F(.,.)$ is a g-pseudomonotone operator.

Taking $v = u$ in (3.2), we get

$$\rho F(\eta_{n+1}, g(u)) + \langle g(u_{n+1}) - g(u_n), g(u) - g(u_{n+1})\rangle + \gamma \|g(u) - g(u_{n+1})\|^2 \geqslant 0.$$

which can be written as

$$\begin{aligned}\langle g(u_{n+1}) - g(u_n), g(u) - g(u_{n+1})\rangle \\ \geqslant -\rho F(\eta_{n+1}, g(u)) - \gamma \|g(u) - g(u_{n+1})\|^2 \\ \geqslant -\gamma \|g(u) - g(u_{n+1})\|^2 - \gamma \|g(u_{n+1}) - g(u)\|^2,\end{aligned} \tag{3.7}$$

where we have used (3.6).

Setting $u = g(u) - g(u_{n+1})$ and $v = g(u_{n+1}) - g(u_n)$ in (2.4), we obtain

$$\begin{aligned}2\langle g(u_{n+1}) - g(u_n), g(u) - g(u_{n+1})\rangle = \|g(u) - g(u_n)\|^2 - \|g(u) - g(u_{n+1})\|^2 \\ - \|g(u_{n+1}) - g(u_n)\|^2.\end{aligned} \tag{3.8}$$

Combining (3.7) and (3.8), we have

$$\{1-\gamma\}\|g(u_{n+1}) - g(u)\|^2 \leqslant \|g(u_n) - g(u)\|^2 - \{1-\gamma\}\|g(u_{n+1}) - g(u_n)\|^2,$$

the required result (3.4). □

THEOREM 3.2. *Let H be a finite dimensional space and let $g: H \longrightarrow H$ be injective. Let $T: H \longrightarrow C(H)$ be M-Lipschitz continuous operator. If $\gamma \leqslant 1$, then the sequence $\{u_n\}_1^\infty$ given by Algorithm* 3.1 *converges to a solution u of* (2.1).

Proof. Let $u \in H$ be a solution of (2.1). From (3.4), it follows that the sequence $\{||g(u) - g(u_n)||\}$ is nonincreasing and consequently $\{g(u_n)\}$ is bounded. Thus it follows that the sequence $\{u\}$ is bounded under the assumptions of g. Furthermore, we have

$$\sum_{n=0}^{\infty}\{1-\gamma\}||g(u_{n+1}) - g(u_n)||^2 \leqslant ||g(u_0) - g(u)||^2,$$

which implies that

$$\lim_{n\to\infty} ||g(u_{n+1}) - g(u_n)|| = 0. \tag{3.9}$$

Let $\hat{u}$ be the limlit point of $\{u_n\}_1^\infty$; a subsequence $\{u_{n_j}\}_1^\infty$ of $\{u_n\}_1^\infty$ converges to $\hat{u} \in H$. Replacing w_n by u_{n_j} in (3.2), taking the limit $n_j \longrightarrow \infty$ and using (3.9), we have

$$F(\hat{\nu}, g(v)) + \gamma \|g(v) - g(\hat{u})\| \geqslant 0, \quad \forall g(v) \in K,$$

which implies that $\hat{u}$ solves the multivalued regularized equilibrium problem (2.1) and

$$\|g(u_{n+1}) - g(u)\|^2 \leqslant \|g(u_n) - g(u)\|^2.$$

Thus, it follows from the above inequality that $\{u_n\}_1^\infty$ has exactly one limit point $\hat{u}$ and

$$\lim_{n\to\infty} g(u_n) = g(\hat{u}).$$

Since g is injective, thus

$$\lim_{n\to\infty} (u_n) = \hat{u}.$$

It remains to show that $\nu \in T(u)$. Using the M-Lipschitz continuity of T, we have

$$||\nu_n - \nu|| \leqslant M(T(u_n), T(u)) \leqslant \delta ||u_n - u||,$$

which implies that $\nu_n \longrightarrow \nu$ as $n \longrightarrow \infty$. Now consider

$$\begin{aligned} \mathrm{d}(\nu, T(u)) &\leqslant ||\nu - \nu_n|| + \mathrm{d}(\nu, T(u)) \\ &\leqslant ||\nu - \nu_n|| + M(T(u_n), T(u)) \\ &\leqslant ||\nu - \nu_n|| + \delta ||u_n - u|| \longrightarrow 0, \quad n \longrightarrow \infty, \end{aligned}$$

where $\mathrm{d}(\nu, T(u)) = \inf\{||\nu - z|| : z \in T(u)\}$ and $\delta > 0$ is the M-Lipschitz continuity constant. From the above inequality, it follows that $\mathrm{d}(\nu, T(u)) = 0$. This implies that $\nu \in T(u)$, since $T(u) \in C(H)$. This completes the proof. □

It is well-known that to implement the proximal point methods, one has to calculate the approximate solution implicitly, which is in itself a difficult problem. To overcome this drawback, we suggest another iterative method, the convergence of which requires only the partially relaxed strongly monotonicity, which is a weaker condition that cocoercivity.

For a given $u \in H : g(u) \in K, \nu \in T(u)$ consider the problem of finding $w \in H : g(w) \in K$ such that

$$\rho F(\nu, g(v)) + \langle g(w) - g(u), g(v) - g(w)\rangle + \gamma \|g(v) - g(w)\|^2 \geqslant 0, \quad \forall g(v) \in K, \tag{3.10}$$

which is also called the auxiliary uniformly regularized equilibrium problem. Note that problems (3.1) and (3.10) are quite different. If $w = u$, then clearly w is a solution of (2.1). This fact enables us to suggest and analyze the following iterative method for solving (2.1).

ALGORITHM 3.2. For a given $u_0 \in K$, compute the approximate solution u_{n+1} by the iterative scheme

$$\rho F(\nu_n, g(v)) + \langle g(u_{n+1}) - g(u_n), g(v) - g(u_{n+1})\rangle$$
$$+\gamma \|g(v) - g(u_{n+1})\|^2 \geqslant 0, \quad \forall g(v) \in K \tag{3.11}$$
$$\eta_n \in T(u_n) : ||\eta_{n+1} - \eta_n|| \leqslant M(T(u_{n+1}), T(u_n)). \tag{3.12}$$

For suitable and appropriate choice of the operators and the spaces, one can obtain a number of iterative methods for solving equilibrium problems and variational inequalities.

We now study the convergence of Algorithm 3.2. The analysis is in the spirit of Theorem 3.1.

THEOREM 3.3. *Let the bifunction $F(.,.)$ be partially relaxed strongly g-monotone with constant $\alpha > 0$. If u_{n+1} is the approximate solution obtained from Algorithm* 3.2 *and $u \in H$ is a solution of* (2.1)*, then*

$$\{1 - \gamma\}\|g(u) - g(u_{n+1})\|^2 \leqslant \|g(u) - g(u_n)\|^2$$
$$-\{1 - 2\rho\alpha - \gamma\}\|g(u_n) - g(u_{n+1})\|^2. \tag{3.13}$$

Proof. Let $u \in H$ be a solution of (2.1). Then

$$F(\nu, g(v)) + \gamma \|g(v) - g(u)\|^2 \geqslant 0, \quad \forall g(v) \in K. \tag{3.14}$$

Taking $v = u_{n+1}$ in (3.14), we have

$$F(\nu, g(u_{n+1})) + \gamma \|g(u_{n+1}) - g(u)\|^2 \geqslant 0. \tag{3.15}$$

Letting $v = u$ in (3.11), we obtain

$$\rho F(\nu_n, g(u)) + \langle g(u_{n+1}) - g(u_n), g(u) - g(u_{n+1})\rangle + \gamma \|g(u) - g(u_{n+1})\|^2 \geqslant 0,$$

which implies that

$$\begin{aligned}
&\langle g(u_{n+1})-g(u_n), g(u)-g(u_{n+1})\rangle \\
&\quad \geqslant -\rho F(v_n, g(u)) - \gamma\|g(u)-g(u_{n+1})\|^2 \\
&\quad \geqslant \rho\{F(v_n, g(u)) + F(v, g(u_{n+1}))\} - \gamma\|g(u)-g(u_{n+1})\|^2 \\
&\qquad -\gamma\|g(u_n)-g(u_{n+1})\|^2 \\
&\quad \geqslant -\alpha\rho\|g(u_n)-g(u_{n+1})\|^2 - \gamma\|g(u)-g(u_{n+1})\|^2 \\
&\qquad -\gamma\|g(u_n)-g(u_{n+1})\|^2. \qquad (3.16)
\end{aligned}$$

since $F(.,.)$ is partially relaxed strongly g-monotone with constant $\alpha > 0$.
Combining (3.8) and (3.16), we obtain the required result (3.13). □

Using essentially the technique of Theorem 3.2, one can study the convergence analysis of Algorithm 3.2.

REMARK 3.1. In this paper, we have studied a new class of equilibrium problems known as the multivalued regularized equilibrium problems. We have shown that the auxiliary principle technique can be extended for solving multivalued regularized equilibrium problems with suitable modifications. We note that this technique is independent of the projection of the operator. Using essentially the technique of this paper, one can suggest and analyze a number of iterative methods for solving mixed multivalued mixed quasi regularized general equilibrium problems.

Acknowledgement

This research is supported by the Higher Education Commission, Pakistan, through research grant No: 1-28/HEC/HRD/2005/90.

References

1. Blum, E. and Oettli, W. (1994), From optimization and variational inequalities to equilibrium problems, *Student Mathematics* 63, 123–145.
2. Clark, F.H., Ledyaev, Y.S., Stern, R.J. and Wolenski, P.R. (1998), *Nonsmooth Analysis and Control Theory,* Springer Verlag, New York.
3. Giannessi, F. and Maugeri, A. (1995), *Variational Inequalities and Network Equilibrium Problems,* Plenum Press, New York.
4. Giannessi, F., Maugeri, A. and Pardalos, P.M. (2001), *Equilibrium Problems: Nonsmooth Optimization and Variational Inequality Models,* Kluwer Academic Publishers, Dordrecht.
5. Glowinski, R. Lions, J.L. and Tremolieres, R. (1981), *Numerical Analysis of Variational inequalities,* North-Holland, Amsterdam.
6. Noor, M.A. (2003), New extragradient-type methods for general variational inequalities, *Journal of Mathematical Analysis and Applications* 277, 379–395.
7. Noor, M.A. (2003), Multivalued general equilibrium problems, *Journal of Mathematical Analysis and Applications* 283, 140–149.

8. Noor, M.A. (2004), Some developments in general variational inequalities, *Applied Mathematics and Computation* 152, 199–277.
9. Noor, M.A., and Oettli, W. (1994), On general nonlinear complementarity problems and quasi-equilibria, *Le Matematiche* 49, 313–331.
10. Patriksson, M. (1999), *Nonlinear Programming and Variational Inequality Approach: A Unified Approach,* Kluwer Academic Publishers, Dordrecht.
11. Poliquin, R.A., Rockafellar, R.T. and Thibault, L. (2000), Local differentiability of distance functions, *Transactions of American Mathematical Society* 352, 5231–5249.
12. Stampacchia, G. (1964), Formes bilineaires coercitives sur les ensembles convexes, *Comptes Rendus de l'Academie des Sciences, Paris* 258, 4413–4416.